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K-ϵ Model : Overview and Fisher equation

- Overview :

  -> Reaction and diffusion system :

$$\partial_t \mathcal{E} = \underbrace{D \nabla^2 \mathcal{E}}_{\text{Diffusion}} + \underbrace{f(\mathcal{E})}_{\text{Reaction}}$$

  -> Turbulence spreading

  - "Minimal problem" : propagation of a patch of turbulence from a region where is locally excited to a region of weaker excitation or even local damping.

$$\frac{\partial \mathcal{E}}{\partial t} - \underbrace{\frac{\partial}{\partial x} D(\mathcal{E}) \frac{\partial \mathcal{E}}{\partial x}}_{\text{Diffusion}} = \underbrace{\gamma(x)\mathcal{E} - \gamma_{NL}\mathcal{E}^2}_{\text{Reaction}}$$

$D(\mathcal{E}) \sim \mathcal{E}$

  - (Above equation) Looks like a Fisher equation :

    Indicates a ballistically expanding front , $c \sim (\gamma D)^{1/2}$

- Derivation

→ Local excitation/damping + Non-local transition $(T(\Delta x, \Delta t))$

⇝ Step size Δx, Δt : radial couplings to mesoscales

→ A Fokker-Planck model of turbulence intensity $\mathcal{E}(x,t)$

$$\mathcal{E}(x, t+\Delta t) = \cancel{\mathcal{E}(x,t)+} [\gamma(x)\mathcal{E}(x) - \gamma_{NL}\mathcal{E}^{\alpha+1}(x)]\Delta t$$

$$+ \int d(\Delta x)\, T(x, \Delta x, \Delta t)\, \mathcal{E}(x-\Delta x, t)$$

$\gamma(x)$: Local excitation / growth rate

$\gamma_{NL}(x)$: Local nonlinear damping rate. $= \gamma_{NL}\mathcal{E}^{\alpha}(x)$, $\frac{1}{2}<\alpha<1$

Weak turbulence ⟺ $\alpha \sim 1$

Strong turbulence ⟺ $\alpha \sim \frac{1}{2}$

$T(x, \Delta x, \Delta t)$: Transition probability . $\underbrace{\Delta x > \Delta x_c \ , \ \Delta t > \tau_c}_{\text{Step size}}$.

Moments of T : $\int d(\Delta x)\, T = 1$ ⟺ conservation of prob.

$$\int d(\Delta x)\, T\Delta x = \langle \Delta x \rangle$$

$$\int d(\Delta x)\, T(\Delta x)^2 = \langle \Delta x \Delta x \rangle$$

Assume $\Delta x \, \mathcal{E}'/\mathcal{E} < 1 \Rightarrow$ expand the nonlocal kernel

step size Δx smaller than gradient scale of fluctuation intensity.

$$\mathcal{E}(x,t) + \Delta t \frac{\partial \mathcal{E}}{\partial t} = [\gamma(x) - \gamma_{NL}\mathcal{E}^{\alpha}]\,\mathcal{E}\,\Delta t + \int d(\Delta x)\, T\, \mathcal{E}$$

$$- \frac{\partial}{\partial x}\int d(\Delta x)\, T\, \Delta x\, \mathcal{E} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\int d(\Delta x)\, T\, (\Delta x)^2 \mathcal{E}$$

$$\rightsquigarrow \boxed{\frac{\partial}{\partial t}\mathcal{E}(x,t) = [\gamma(x) - \gamma_{NL}\mathcal{E}^{\alpha}]\,\mathcal{E}(x,t) - \frac{\partial}{\partial x}[V_{\mathcal{E}}\mathcal{E}(x,t)] + \frac{\partial^2}{\partial x^2}[D_{\mathcal{E}}(x)\mathcal{E}(x,t)]}$$

$V_{\mathcal{E}} = \left\langle \frac{\Delta x}{\Delta t} \right\rangle$, $D_{\mathcal{E}} = \left\langle \frac{(\Delta x)^2}{2\Delta t} \right\rangle$

Drift velocity ... Intensity diffusivity

$V_{\mathcal{E}}$, $D_{\mathcal{E}}$ ~ $\mathcal{E}(x,t)$ dependent.

Fokker-Planck eqn for "coarse-grained" turbulence energy density. (on scale of Δ_c)

Question : How to calculate $V_{\mathcal{E}}$, $D_{\mathcal{E}}$?

Answer : Wave kinetic eqn.

Wave population density is conserved along ray trajectory.

$$\Rightarrow \frac{\partial N}{\partial t} + (\underline{V}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial \underline{x}}(\omega + \underline{k}\cdot\underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = \underbrace{\gamma_{NC} N}_{\text{Nonpopulation density conserving processes}}$$

$$\frac{d\underline{x}}{dt} = \underline{V}_{g} + \underline{V} \,, \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}}(\omega + \underline{k}\cdot\underline{V})$$

$\underline{V}_g$: wave group velocity

$\underline{V}$: local flow velocity

$$\frac{dx}{dt} = \underbrace{V_{gr}}_{\text{radial group velocity}} + \underbrace{\langle V_r \rangle}_{\text{mean radial flow}} + \underbrace{\delta V_r}_{\text{fluctuating flow.}}$$

$$D = \int_{-\infty}^{\infty} d\tau \langle \delta V_r(0) \delta V_r(\tau) \rangle \cong D_0 \varepsilon^{\alpha}$$

Assume : No large scale coherent flow is present.

$\Downarrow$

Neglect $\langle V_r \rangle$

$\Downarrow$

$$V_{\varepsilon} = V_{gr} + V_{drift} \,. \quad V_{drift} = \frac{\partial}{\partial x}(D_0 \varepsilon^{\alpha})$$

$\Downarrow$

$$\begin{cases} V_{\varepsilon} = V_{gr} + \frac{\partial}{\partial x}(D_0 \varepsilon^{\alpha}) \\ D_{\varepsilon} = D_0 \varepsilon^{\alpha} \end{cases}$$

Plug V_{ε}, D_{ε} into the F-P eqn.

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x}\left(V_{gr}\,\mathcal{E}\right) - \frac{\partial}{\partial x} D_0 \mathcal{E}^{\alpha} \frac{\partial \mathcal{E}}{\partial x} = \left(\gamma(x) - \gamma_{NL}\,\mathcal{E}^{\alpha}\right)\mathcal{E}$$

The effect of V_{gr} can be cancelled by a Galilean transformation.

Thus for the case $\gamma = \text{const}$, $\gamma_{NL} = \text{const}$, $D_0 = \text{const}$, $\alpha = 1$

re-scale : $x \to (\gamma_{NL}/2D_0)^{1/2}\, x$, $t \to \gamma t$

$\mathcal{E} \to (\gamma_{NL}/\gamma)\,\mathcal{E}$

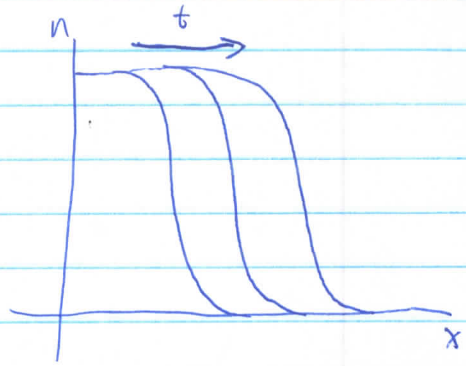
The above eqn becomes

$$\frac{\partial \mathcal{E}}{\partial t} - \frac{1}{4}\frac{\partial^2}{\partial x^2}\mathcal{E}^2 - \mathcal{E}(1-\mathcal{E}) = 0$$

Variant of Fisher - Kolmogorov - Petrovski - Piskunov (Fisher-KPP) eqn.

Proto-type of Fisher-KPP eqn:

$$\frac{\partial n}{\partial t} - D\partial_x^2 n = \gamma n(1-n)$$



travelling wave solution

For the wave tip where $n \sim \tilde{n} \ll 1$

$$\frac{\partial \tilde{n}}{\partial t} - D\partial_x^2 \tilde{n} = \gamma \tilde{n}$$

Impose $\tilde{n} = \tilde{n}(x-ct)$, it becomes

$$-c\frac{\partial \tilde{n}}{\partial x} - D\frac{\partial^2 \tilde{n}}{\partial x^2} = \gamma \tilde{n}$$

$$\tilde{n} \sim n_0 \exp[-\alpha(x-ct)]$$

$$\Rightarrow \quad \alpha c\tilde{n} - D\alpha^2 \tilde{n} = \gamma \tilde{n}$$

$$\alpha = \frac{c}{2D} \pm \frac{1}{2}\left(\frac{c^2}{D^2} - \frac{4\gamma}{D}\right)^{1/2}$$

$C_{min} = 2(\gamma D)^{1/2}$ $\leftrightarrow$ selected speed.

Faced with "Speed Selection"!

$\boxed{c < C_{min}}$ $\Rightarrow$ α is complex, the front oscillates $\Rightarrow$ $\boxed{\text{unstable}}$

So $c \geq C_{min}$

Further, would all fronts with speeds $c \geq c_{min}$ survive? No!

Front speed is selected according to the steep stiffness of the initial condition, $\tilde{n}(x,0) = e^{-\lambda x}$, λ: steep stiffness

Many have discussed about front stability of Fisher-KPP equ.

We use the results by M. R. Evans,

① If the initial profile decays faster than $\tilde{n}(x,0) \sim e^{-\alpha_{min} x}$,

where $\alpha_{min} = \frac{c_{min}}{2D}$ (which also means $\lambda > \alpha_{min}$)

then wave travels at $c_{min} = 2(\gamma D)^{\frac{1}{2}}$.

② If the initial profile decays less steeply, $\lambda < \alpha_{min}$,

then wave travels at a faster speed $v(\lambda) = D\lambda + \frac{\gamma}{\lambda}$

In the case of turbulence spreading, here we assume the front is steep. Thus, fronts travel at $c_{min} = 2(\gamma D)^{\frac{1}{2}}$

Go back to the Fokker-Planck model, neglecting effect of V_{gr}.

$$\frac{\partial \mathcal{E}}{\partial t} - \frac{\partial}{\partial x} D_0 \mathcal{E} \frac{\partial \mathcal{E}}{\partial x} = \gamma \mathcal{E} - \gamma_{NL} \mathcal{E}^2 .$$

Argue $\mathcal{E} \sim \frac{\gamma}{\gamma_{NL}}$, using Fisher-KPP's conclusion.

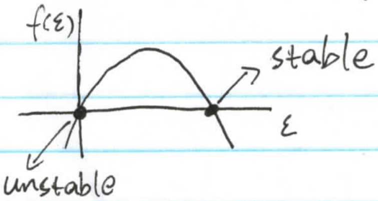
where $D \sim D_0 \mathcal{E} \sim D_0 \frac{\gamma}{\gamma_{NL}}$. $\quad C_{eff} \sim 2 (\gamma D)^{\frac{1}{2}} \sim 2 \left(\frac{D_0 \gamma^2}{\gamma_{NL}} \right)^{\frac{1}{2}}$

↑

Turbulence spreading speed.

- From Fisher to Fitzhugh-Nagumo, $\partial_t \mathcal{E} - D \nabla^2 \mathcal{E} = f(\mathcal{E})$ ↖ Reaction

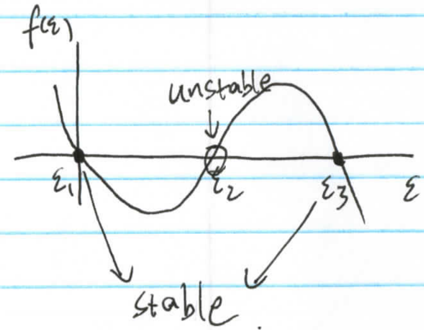
  -> Fisher : unistable



$$f(\mathcal{E}) = \gamma \mathcal{E} - \gamma_{NL} \mathcal{E}^2$$

  -> Nagumo : bi-stable

$$f(\mathcal{E}) = A (\mathcal{E} - \mathcal{E}_1)(\mathcal{E}_2 - \mathcal{E})(\mathcal{E} - \mathcal{E}_3)$$



References :

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[2] J. Venegas-Ortiz, R.J. Allen and Martin R. Evans, Genetics 113, 158642 (2013)